# Structured Multigrid for Taylor-Hood Finite Elements

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# Overview

1. Stokes Equations
2. Geometric Setup
3. GMRES and Multigrid
4. First results
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6. Future Work

# Stokes Equations

$$-\nabla \cdot (\nu \epsilon(\mathbf{u})) + \nabla p = \mathbf{f} \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (2)$$

$\rightarrow$ assuming $\nu$ to be constant simplifies (1) to

$$-\nu \nabla^2 \mathbf{u} + \nabla p = \mathbf{f}$$

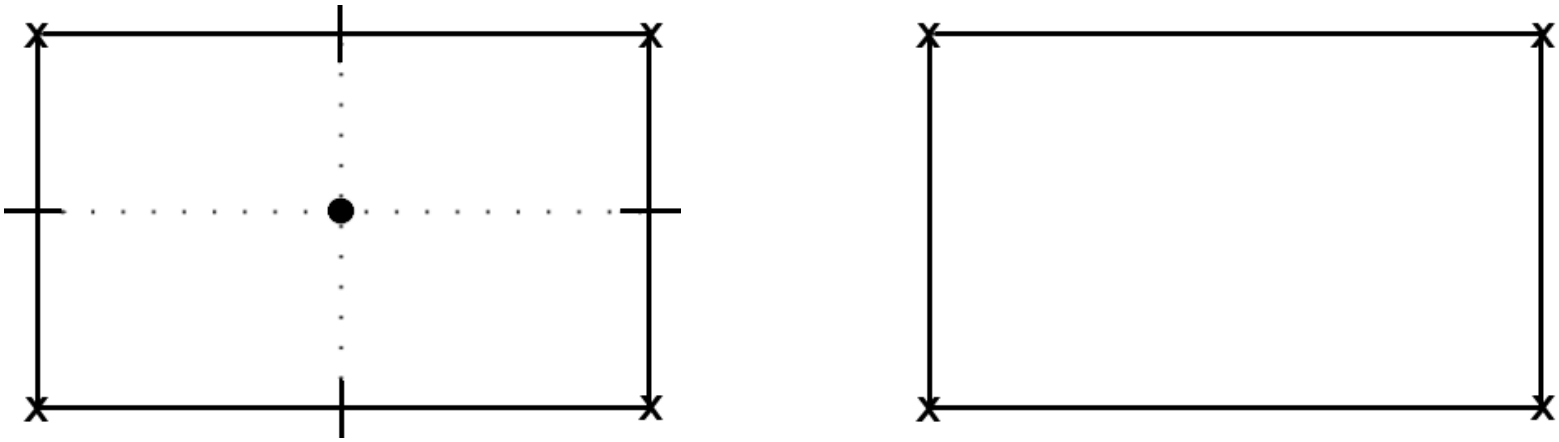
Weak form: Find $u \in H_0^1(\Omega)$ and $p \in L^2(\Omega)/\mathbb{R}$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega) \\ b(\mathbf{u}, q) &= 0 \qquad \forall q \in L^2(\Omega)/\mathbb{R} \end{aligned}$$

$$\Rightarrow \begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix}$$

# TAYLOR-HOOD ELEMENTS

Q2 elements for velocity - Q1 elements for pressure



$\rightarrow$ known to be inf-sup stable

$\rightarrow$ equally spaced, regular structure

# Stencil extraction

Number of non-zeros per row (Laplacian):

| | nodes | x-edges | y-edges | centers | **Total** |
|---|---|---|---|---|---|
| nodes | 9 | 6 | 6 | 4 | **25** |
| x-edges | 6 | 3 | 4 | 2 | **15** |
| y-edges | 6 | 4 | 3 | 2 | **15** |
| centers | 4 | 2 | 2 | 1 | **9** |

$\rightarrow$ Natural consequence: localised stencil-based calculations

$\rightarrow$ Stencils independent from each other $\Rightarrow$ parallelisable

# GMRES

Initial system

$$A\mathbf{x} = \mathbf{b}$$

GMRES minimises *2-norm* of residual

$$||\mathbf{r_m}||_2 = ||\mathbf{b} - A\mathbf{x_m}||_2$$

over all vectors in the *Krylov-subspace*

$$\mathbf{x_0} + \mathcal{K}_m(A, \mathbf{r_0}) = \mathbf{x_0} + span\{\mathbf{r_0}, A\mathbf{r_0}, ..., A^{m-1}\mathbf{r_0}\}.$$

(Right-)preconditioning:

$$AM^{-1}\mathbf{u} = \mathbf{b}$$
$$\mathbf{u} = M\mathbf{x}$$

$\rightarrow$ V-Cycle with rediscretisation on coarser grids

# Braess-Sarazin

"Ideal" Braess-Sarazin update:

$$\begin{bmatrix}\mathbf{u}\\ p\end{bmatrix}^{new} = \begin{bmatrix}\mathbf{u}\\ p\end{bmatrix}^{old} + \omega_{BS} \begin{bmatrix}tD & B\\ B^T & 0\end{bmatrix}^{-1} \left( \begin{bmatrix}f\\ g\end{bmatrix} - A \begin{bmatrix}\mathbf{u}\\ p\end{bmatrix}^{old} \right)$$

$\rightarrow$ factorised system

$$\begin{bmatrix}tD & 0\\ B^T & S\end{bmatrix} \begin{bmatrix}I & \frac{1}{t}D^{-1}B\\ 0 & I\end{bmatrix} \begin{bmatrix}\delta\mathbf{u}\\ \delta p\end{bmatrix} = \begin{bmatrix}\mathbf{r_u}\\ r_p\end{bmatrix}$$

with Schur complement

$$S = -\frac{1}{t}B^T D^{-1} B$$

## BRAESS-SARAZIN

→ compute $S$ exactly

→ approximately solve for solution using

$$S\delta p = r_p - \frac{1}{t} B^T D^{-1} \mathbf{r_u} \tag{3}$$

$$\delta \mathbf{u} = \frac{1}{t} D^{-1} (\mathbf{r_u} - B\delta p). \tag{4}$$

with one sweep of weighted Jacobi on (3), plugging solution into (4).

→ parameter choice: $t = 1.1, \omega = 0.7$
(based on small experiments)

# Fine-Scale Parallelism

| | |
|---|---|
| GMRES: | matrix-vector and vector-vector calculations<br>$\rightarrow$ easy parallelisation |
| Multigrid: | matrix-vector and vector-vector calculations<br>$\rightarrow$ easy parallelisation |
| | interpolation/restriction<br>$\rightarrow$ easy parallelisation |
| Braess-Sarazin: | matrix-vector and vector-vector calculations<br>$\rightarrow$ easy parallelisation |
| Solve with $S$: | weighted Jacobi<br>$\rightarrow$ easy parallelisation |
| Calculate $\delta\mathbf{u}$: | matrix-vector and vector-vector calculations<br>$\rightarrow$ easy parallelisation |

# GPU CONSIDERATIONS

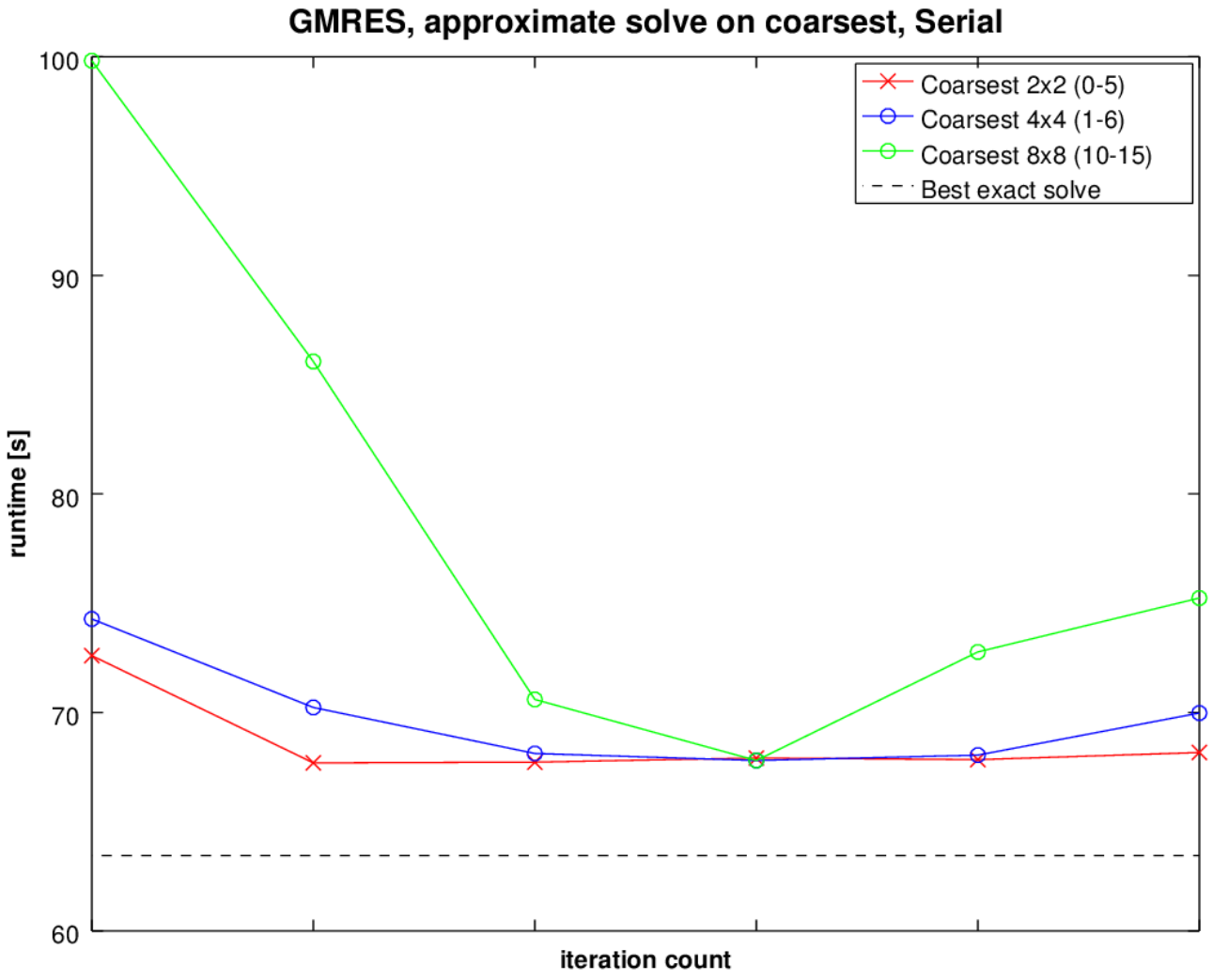
- fine-grain parallelism perfect for GPU's
- implemented using OpenCL
- additional cost of communication
  $\rightarrow$ moving initial data to GPU('s)
  $\rightarrow$ communication between GPU's
- Galerkin coarse-grid operators expensive
- direct solver on GPU?

# Remarks on Numerical Experiments

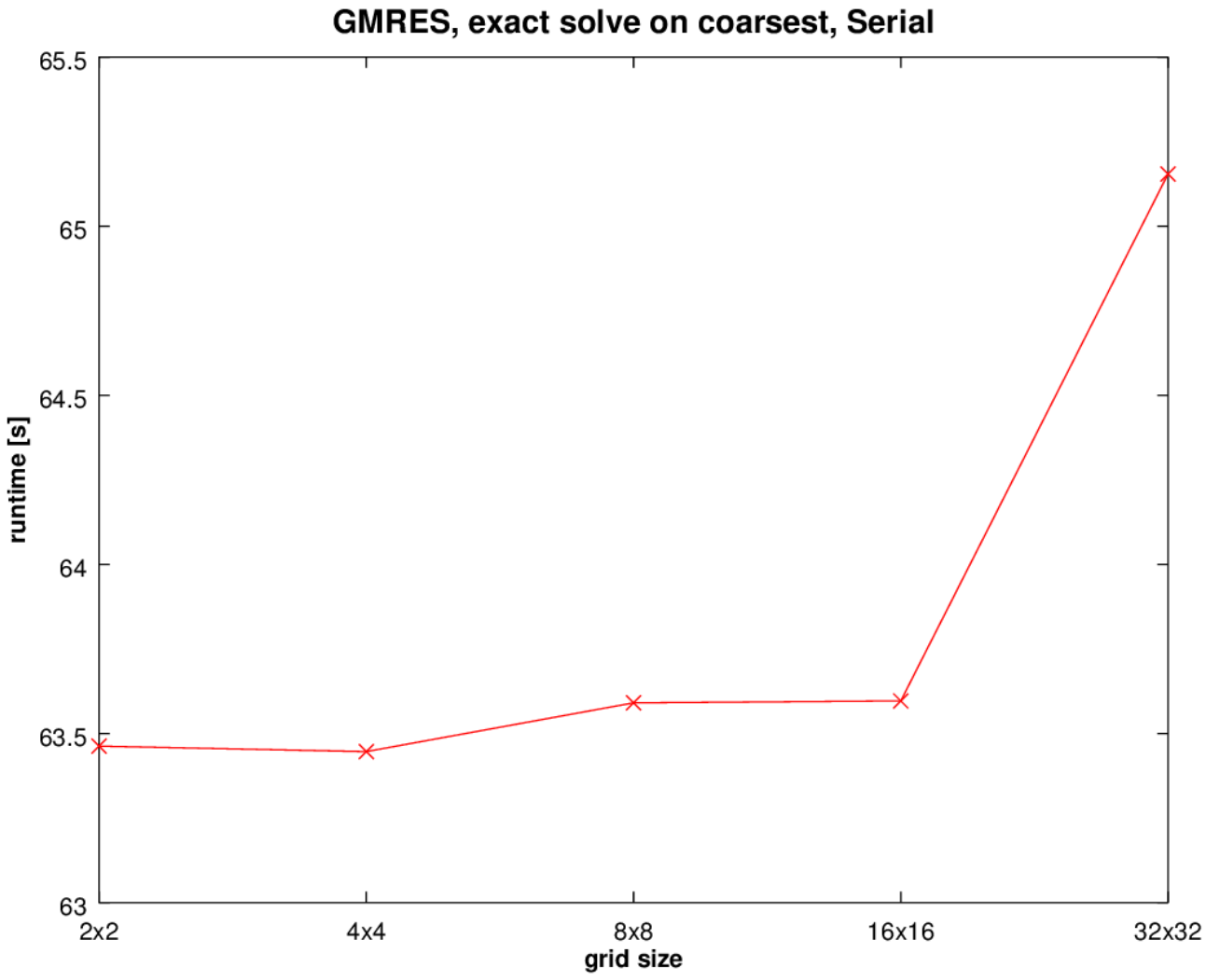
- VCycle preconditioner

  possible solve on coarsest grid:
  1. approximate: Braess-Sarazin smoother
  2. exact: UMFPACK library on CPU
- GPU/CPU calculations identical
  (up to floating-point error)
- CPU: Intel Xeon; GPU: NVIDIA Tesla K20X
- 512x512 element patch: about 2.25M degrees of freedeom
  (comparison: about 0.25M for Poisson)

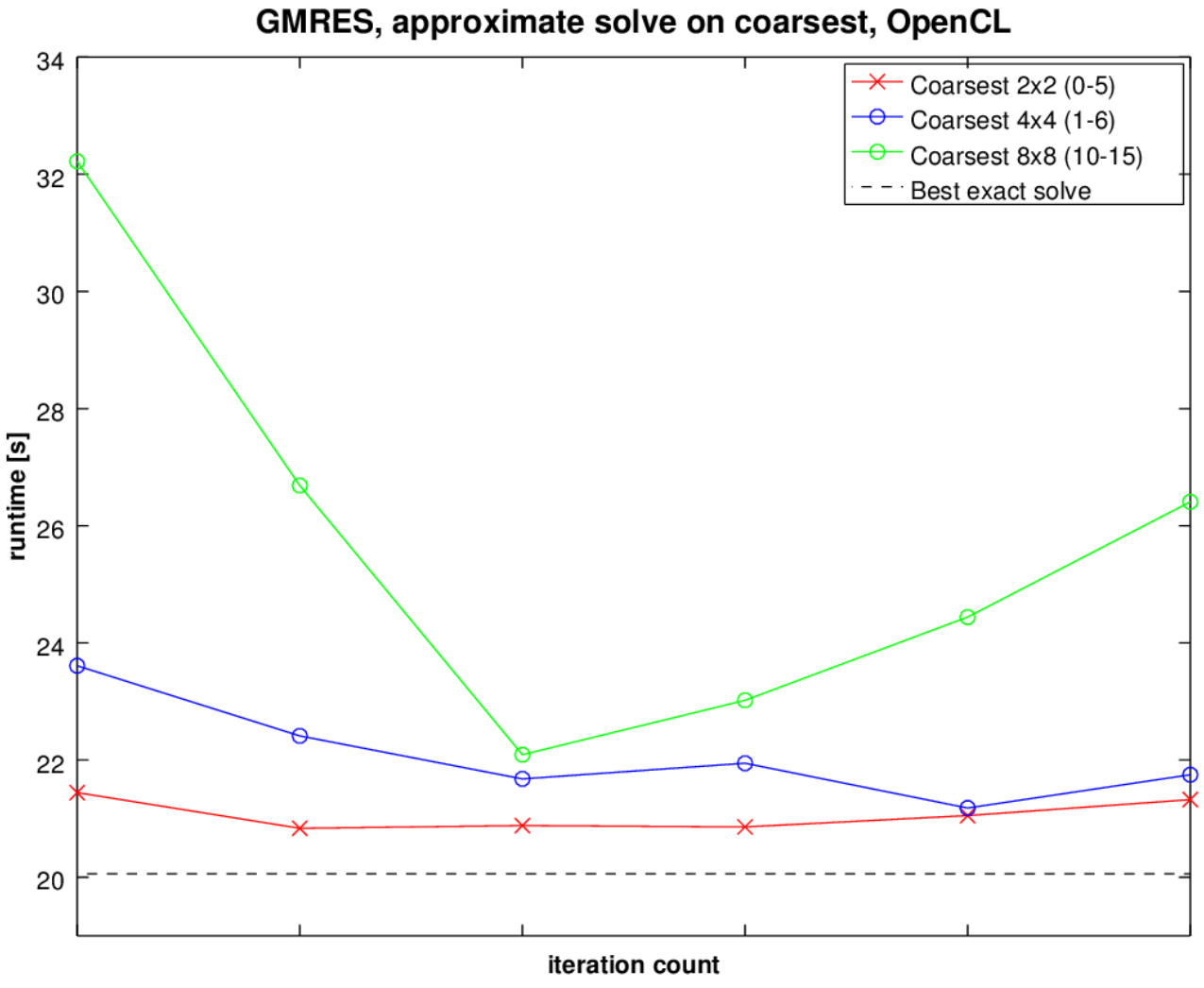
# Serial performance



# Serial performance



# OpenCL performance



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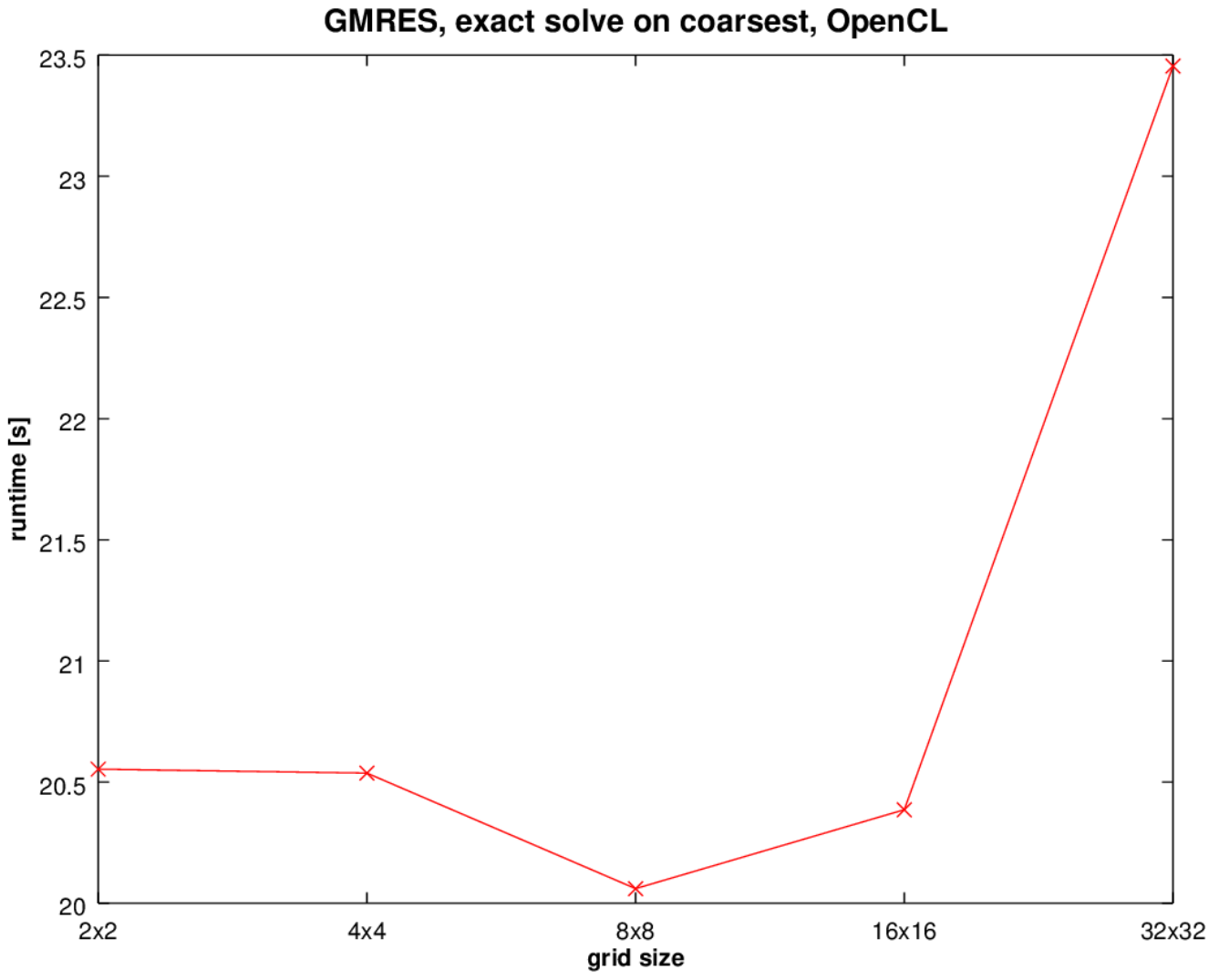
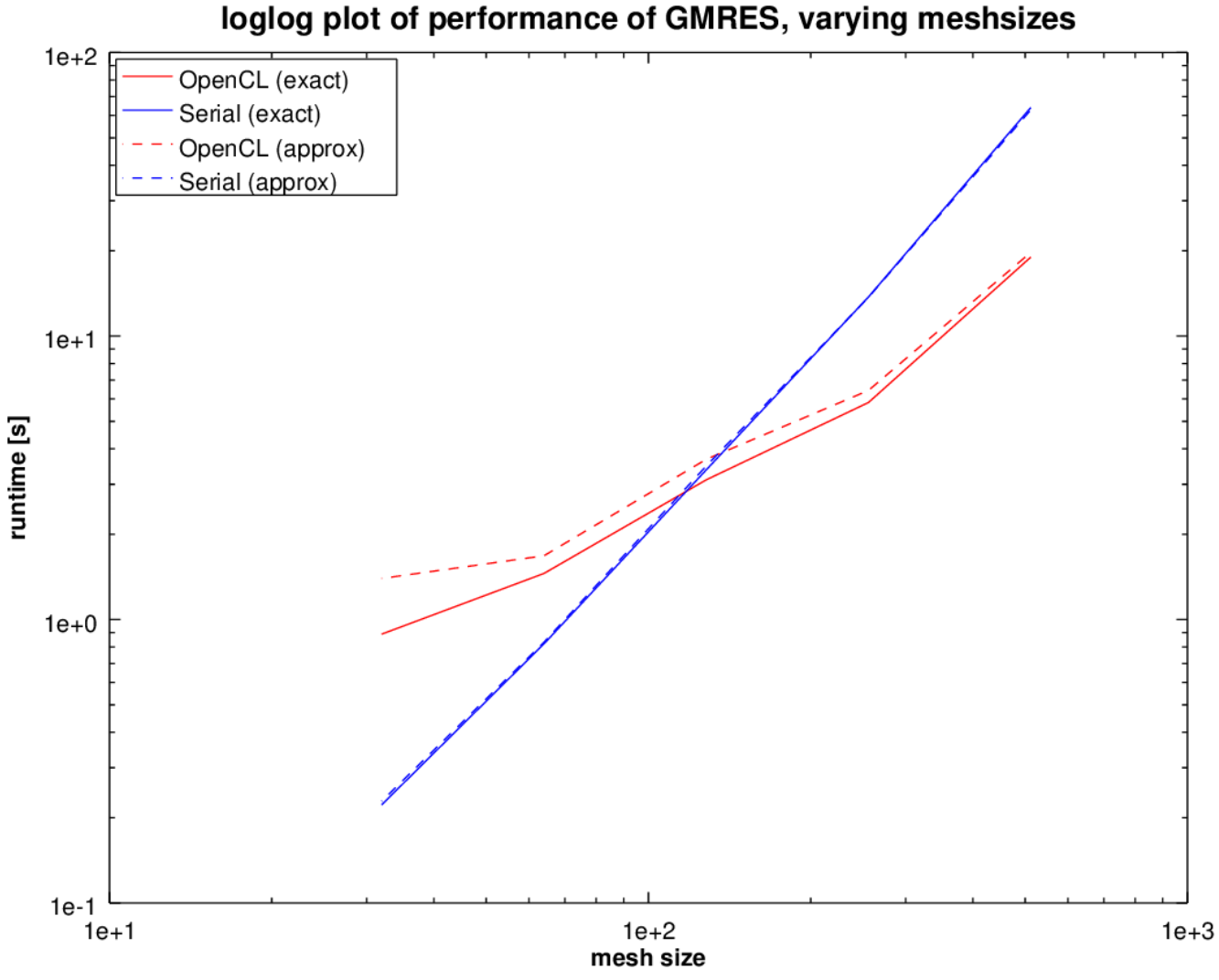


# SPEEDUP COMPARISON



# Summary

- structured grid
  $\rightarrow$ stencil formulation
- low memory cost
- many independent calculations
  $\rightarrow$ well parallelisable
- exact solve on coarsest grid marginally more efficient than approximated solve
- OpenCL speeds things up by a factor of up to 3 on larger meshes
- no speed-up on small meshsizes
  $\rightarrow$ overhead of moving to GPU dominant

# Future Work

- Adding MPI to the mix
  - Goal: Taking advantage of heterogeneous systems
  - Plan: MPI-based partitioning with one GPU per MPI thread
  - Complications in communication:
    GPU $\rightarrow$ MPI $\rightarrow$ MPI $\rightarrow$ GPU
- Vanka relaxation

  Slow in serial (Braess-Sarazin better choice), potential benefits in parallel
- efficient Galerkin coarsening on GPU
- direct solve on GPU?

# Thank you!